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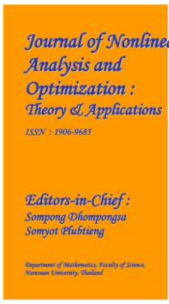

RECURRENCE RELATIONS RELATED TO METALLIC RATIOS

B. Ramesh, Research Scholar, Postgraduate and Research Department of Mathematics
Dwaraka Doss Goverdhan Doss Vaishnav College, Arumbakkam, Chennai – 600 106
Email: jeeyaram@gmail.com
Dr. R. Sivaraman Associate Professor, Postgraduate and Research Department of Mathematics
Dwaraka Doss Goverdhan Doss Vaishnav College, Arumbakkam, Chennai – 600 106
Email: rsivaraman1729@yahoo.co.in

ABSTRACT
Recurrence Relations are useful in exploring several interesting properties and determine patterns that occur among numbers. In this paper, we have introduced a general linear recurrence relation using two parameters as coefficients. We have obtained Binet's formula for this general recurrence relation and had obtained nice results regarding the limiting ratios corresponding to the successive terms formed through the recurrence relation considered. Finally, we have demonstrated the connection of recurrence relation with the metallic ratios of order *k*.

Key words: Recurrence Relation, Binet's Formula, Limiting Ratio. Metallic Ratio.

1. INTRODUCTION
Ever since the concept of Fibonacci sequence had its great impact in almost all fields of knowledge, mathematicians have been trying to generalize the recurrence relation regarding the formation of Fibonacci sequence producing other amusing sequences. Each of such sequences has their own applications and importance. In this paper, we try to solve the linear recurrence relation of special type and obtain metallic ratios as limiting ratio.

2. DEFINITION
2.1 The linear recurrence relation is defined by

$$p\,(n+2) = \lambda_1 p\,(n+1) + \lambda_2 p(n), n \geq 0, p(0) = 0, p(1) = 1 \qquad (1)$$

Note that λ_1 and λ_2 are any two real numbers.

2.2 The ratio of $(n + 1)^{th}$ term to the n^{th} term of a sequence as $n \to \infty$ is defined as limiting ratio of that sequence.

That is, the limiting ratio of the sequence satisfying (1) is given by $\lim_{n\to\infty} \frac{p(n+1)}{p(n)}$

3. Theorem 1
If λ_1 and λ_2 are two real numbers such that $p(n + 2) = \lambda_1 p(n + 1) + \lambda_2 p(n), n \geq 0$, where $p(0) = 0, p(1) = 1$, ${\lambda_1}^2 + 4\lambda_2 > 0$, then the general solution of the recurrence relation (1) is given by $p(n) = \frac{m_1^n - m_2^n}{m_1 - m_2}$ (2)

Proof: The linear recurrence relation is defined by
$p(n+2) = \lambda_1 p(n+1) + \lambda_2 p(n), n \geq 0, p(0) = 0, p(1) = 1$ where λ_1 and λ_2 are two real numbers.

The Characteristic equation is given by $m^2 - \lambda_1 m - \lambda_2 = 0$

The solution of above quadratic equation is $m = \frac{\lambda_1 \pm \sqrt{{\lambda_1}^2 + 4\,\lambda_2}}{2}$

We consider $m_1 = \frac{\lambda_1 + \sqrt{{\lambda_1}^2 + 4\,\lambda_2}}{2}$ and $m_2 = \frac{\lambda_1 - \sqrt{{\lambda_1}^2 + 4\,\lambda_2}}{2}$

Since ${\lambda_1}^2 + 4\,\lambda_2 > 0$, it follows that m_1, m_2 are real roots and $m_1 > m_2$

The general solution is $p(n) = Am_1^n + Bm_2^n$.

Apply that initial condition $p(0) = 0$, $A + B = 0$

We get $B = -A$ and from $P(1) = 1$, we have $1 = Am_1 + Bm_2$

Hence $A = \frac{1}{m_1 - m_2}$ and $B = \frac{-1}{m_1 - m_2}$

Thus, $p(n) = \frac{m_1^n - m_2^n}{m_1 - m_2}$ proving (2).

This completes the proof.

Note that the expression for $p(n)$ derived above namely $p(n) = \frac{m_1^n - m_2^n}{m_1 - m_2}$ is known as Binet's formula.

4. Theorem 2

If λ_1 and λ_2 are two real numbers such that $p(n+2) = \lambda_1\, p(n+1) + \lambda_2\, p(n), n \geq 0,$ $p(0) = 0, p(1) = 1$ then the limiting ratio of $p(n)$ is $m_1 = \frac{\lambda_1 + \sqrt{{\lambda_1}^2 + 4\,\lambda_2}}{2}$ (3)

Proof: From the theorem 1, $p(n) = \frac{m_1^n - m_2^n}{m_1 - m_2}$

$$\text{So, } p(n+1) = \frac{m_1^{n+1} - m_2^{n+1}}{m_1 - m_2}$$

$$\frac{p(n+1)}{p(n)} = \frac{m_1^{n+1} - m_2^{n+1}}{m_1^n - m_2^n}$$

$$\frac{p(n+1)}{p(n)} = \frac{m_1^{n+1}\left(1 - \left(\frac{m_2}{m_1}\right)^{n+1}\right)}{m_1^n\left(1 - \left(\frac{m_2}{m_1}\right)^n\right)}$$

Since $-1 < \frac{m_2}{m_1} < 1$, we get $\left(\frac{m_2}{m_1}\right)^{n+1} \to 0, \left(\frac{m_2}{m_1}\right)^n \to 0$ as $n \to \infty$

$\lim\limits_{n\to\infty} \frac{p(n+1)}{p(n)} = \lim\limits_{n\to\infty} \frac{m_1^{n+1}}{m_1^n} = m_1.$

Using the value of m_1 obtained in Theorem 1, we get

$\lim\limits_{n\to\infty} \frac{p(n+1)}{p(n)} = m_1 = \frac{\lambda_1 + \sqrt{{\lambda_1}^2 + 4\,\lambda_2}}{2}$

This completes the proof.

We now consider certain special cases of interest.

5. SPECIAL CASES:

5.1 When $\lambda_1 = k , \lambda_2 = 1 , {\lambda_1}^2 + 4\lambda_2 = k^2 + 4 > 0$
If $\lambda_1 = k , \lambda_2 = 1$ then according to (3), the limiting ratio would be

$$\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = \frac{k+\sqrt{k^2+4}}{2} \quad (4)$$

The above value is called Metallic Ratio of order k.

5.2 When $\lambda_1 = 1 , \lambda_2 = 1, {\lambda_1}^2 + 4\lambda_2 = 5 > 0$
If $\lambda_1 = 1 , \lambda_2 = 1$, then according to (3), the limiting ratio would be

$$\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = \frac{1+\sqrt{5}}{2} \quad (5)$$

The above value is called Golden Ratio.
5.3 When $\lambda_1 = 2 , \lambda_2 = 1, {\lambda_1}^2 + 4\lambda_2 = 8 > 0$
If $\lambda_1 = 2 , \lambda_2 = 1$, then according to (3), the limiting ratio would be

$$\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = \frac{2+\sqrt{8}}{2} = 1+\sqrt{2} \quad (6)$$

The above value is called Silver Ratio
5.4 When $\lambda_1 = 3,\ \lambda_2 = 1, {\lambda_1}^2 + 4\lambda_2 = 13 > 0$
If $\lambda_1 = 3,\ \lambda_2 = 1$ then according to (3), the limiting ratio would be

$$\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = \frac{3+\sqrt{13}}{2} \quad (7)$$

The above value is called Bronze Ratio.
The Golden ratio, Silver ratio and Bronze ratio are Metallic ratios of order $k = 1,2,3$ respectively.

CONCLUSION

In this paper, we have considered solving a particular Linear Recurrence Relation given in (1) and obtained its closed solution in Binet's Formula version in Theorem 1. In theorem 2, we proved that the limiting ratio of the linear recurrence relation (1) is the positive root of the characteristic equation of (1) namely $m_1 = \frac{\lambda_1 + \sqrt{{\lambda_1}^2 + 4\lambda_2}}{2}$. Using this value, we have discussed certain special cases in which the limiting ratios turns out to be Golden Ratio, Silver Ratio and Bronze Ratio which are first three orders of general metallic ratios. These basic results will help us to understand the behavior of recurrence relation (1) and help us to discuss similar recurrence relations whose solutions may provide a new insight of understanding the asymptotic behaviors of various sequences.

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